Liouville 2D gravity

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May 2020

In the paper, a very simple example of quantum gravity theory is discussed. The simplicity originates from two-dimensional space-time and conformal symmetry. Constructed Liouville theory in classical case is simple and describes 2D spaces of constant curvature. However, the quantization of the metric tensor as an independent field is possible in such a framework. In order to calculate Quantum Field Theory objects, such as correlation functions and partition function, the Polyakov approach may be applied. The simplicity of calculations provides a clear first take on the full quantum theory of gravity, which is still to be constructed.

Basic properties of General Relativity and Conformal Field Theory are mentioned, although basic knowledge of these theories is necessary. The focus is set on the explicit construction of the Liouville action and possible means of quantizing it.

1 General Relativity

General Relativity is one of the most established classical theories, it describes the relation between geometrical deformation of space-time and gravitational forces present in the Universe. The consideration of GR is based on the ansatz, that physical reality remains unchanged under a general transformation of the reference frame. Tensor formalism is particularly useful in the mathematical formulation of such an assumption.

1.1 Diffeomorphism invariance

The invariance under coordinate transformation is usually [2] described in terms of differential geometry. Space-time is considered to be a general pseudo-Riemannian manifold and the transformations are diffeomorphisms. In the following, the structure of the spaces will not be discussed and differential geometry language is unnecessary.
Consider a metric $g_{\mu\nu}$ on pseudo-Riemannian manifold and a diffeomorphism given by the coordinate transformation: $x \to x'$. The metric tensor transforms in a way rank 2 covariant tensor does:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x).$$

The determinant of $g'(x')$ takes form:

$$g' = \det\Big(\frac{\partial x}{\partial x'}\Big)^2 g,$$

hence the transformation of the square root of the determinant of the metric tensor is given by:

$$\sqrt{g} \to \Big|\det\Big(\frac{\partial x}{\partial x'}\Big)\Big|\sqrt{g}.$$

One may now see, that diffeomorphism invariant integral measure is:

$$\sqrt{g}d^D x \to \Big|\det\Big(\frac{\partial x}{\partial x'}\Big)\Big| \cdot \Big|\det\Big(\frac{\partial x'}{\partial x}\Big)\Big|\sqrt{g}d^D x = \sqrt{g}d^D x,$$

Where the second determinant is simply the Jacobian. General Relativity is invariant under change of the reference frame. The action of the theories in curved space-time consists of fully contracted tensors and above invariant measure.

1.2 Tensors of GR

The coordinate invariant structure of General Relativity relies heavily on tensorial notation. The core tensors of the theory are briefly described in this section. Consider a vector field $\vec{v}$, which may be expressed in a set of basis vectors $\hat{e}_i$ as $\vec{v} = v^m \hat{e}_m$. We may calculate how does the vector field $\vec{v}$ change with the change of coordinate components x_i. What is important, we do not assume constant basis vectors:

$$\frac{\partial \vec{v}}{\partial x_i} = \frac{\partial}{\partial x_i}(v^m \hat{e}_m) = \frac{\partial v^m}{\partial x_i}\hat{e}_m + v^m \frac{\partial \hat{e}_m}{\partial x_i} = \Big(\frac{\partial v^m}{\partial x_i} + \Gamma^m{}_{ik} v^k\Big)\hat{e}_m. \tag{1}$$

The components $\Gamma^m{}_{ik} v^k$ are called *Christoffel symbols*, and may be understood as expansion components of the new basis vectors $\frac{\partial \hat{e}_i}{\partial x_k}$ in the old basis:

$$\frac{\partial \hat{e}_i}{\partial x_k} = \Gamma^m{}_{ik}\hat{e}_m.$$

Usually Christoffel symbols are computed thanks to identity:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}\Big(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}\Big). \tag{2}$$

They are closely connected to the *covariant derivative*, which is introduced to include the change of basis vector in (1), hence the definition:

$$\nabla_i v^m = \frac{\partial v^m}{\partial x_i} + \Gamma^m{}_{ik} v^k. \tag{3}$$

In Appendix we show, that the partial derivative of a vector field does not transform as a tensor, however covariant derivative does. This extends to higher rank tensors, in particular the covariant derivative for rank (2,0) tensor is given by:

$$\tau^{ab}{}_{;c} = \partial_c \tau^{ab} + \Gamma^a{}_{cd}\tau^{db} + \Gamma^b{}_{cd}\tau^{da}. \tag{4}$$

The central object in General Relativity is the *Riemann tensor*, which contains information about curvature of a given manifold. It is constructed of metric tensor and its first and second derivatives:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}.$$

In 4D it has 20 independent components, while in 2D there is only 1 independent component [1]. Important symmetries of Riemann tensor are:

- *Skew symmetry*, Riemann tensor is antisymmetric in $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ indices:

$$R_{abcd} = -R_{bacd} = -R_{abdc}$$

- *Interchange symmetry*, symmetric in pairs $\{1,2\} \leftrightarrow \{3,4\}$:

$$R_{abcd} = R_{cdab}$$

- *First Bianchi Identity*, cyclic sum of $\{2,3,4\}$ is qual to 0:

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

- *Second Bianchi Identity.* cyclic sum of covariant derivatives is equal to zero:

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0$$

Ricci tensor is a contracted Riemann tensor and describes how much a given space diverges from a flat one

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}.$$

Ricci scalar is another contraction of the Riemann tensor:

$$R = R^\mu_\mu.$$

Convenient way of calculating the Ricci scalar:

$$R = g^{\mu\nu}\Big(\Gamma^\rho{}_{\mu\nu,\rho} - \Gamma^\rho{}_{\mu\rho,\nu} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\rho{}_{\rho\sigma} - \Gamma^\sigma{}_{\mu\rho}\Gamma^\rho{}_{\nu\sigma}\Big). \tag{5}$$

1.3 Energy momentum tensor conservation

The curvature of space-time is intimately connected with the distribution of mass and energy. From classical field theory, it is well known, information of the energy of a given system is encoded in energy-momentum tensor. Usually, it is understood as Noether current conserved under space-time translations [3]. In the curved space-time the energy momentum tensor may be defined, via variation of the action under metric transformation:

$$\delta S := \frac{1}{2}\int d^D x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu},$$
$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \delta g_{\mu\nu}(x).$$

Under infinitesimal translation $x^\mu \to x^\mu + \epsilon^\mu(x)$ the variation of the metric tensor is:

$$\begin{aligned}\delta g_{\mu\nu} &= \left(\delta^\mu_\nu - \partial_\mu\epsilon^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu\epsilon^\beta\right)g_{\alpha\beta}(x+\epsilon) - g_{\mu\nu}(x)\\ &= g_{\mu\nu}(x+\epsilon) - g_{\mu\nu}(x) - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu\\ &= \epsilon^\alpha\partial_\alpha g_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu.\end{aligned}$$

We shall now prove that $\epsilon^\alpha\partial_\alpha g_{\mu\nu} - \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu$. Starting with the definition of covariant derivative we evaluate $\nabla_\mu\epsilon_\nu$:

$$\nabla_\mu\epsilon_\nu = \nabla_\mu(g_{\mu\alpha}\epsilon^\alpha) = g_{\nu\alpha}\nabla_\mu\epsilon^\alpha = g_{\nu\alpha}(\partial_\mu\epsilon^\alpha + \Gamma^\alpha_{\kappa\mu}\epsilon^\kappa)$$
$$\nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu = g_{\nu\alpha}\partial_\mu\epsilon^\alpha + g_{\mu\alpha}\partial_\nu\epsilon^\alpha + \left(g_{\nu\alpha}\Gamma^\alpha_{\kappa\mu} + g_{\mu\alpha}\Gamma^\alpha_{\kappa\nu}\right)\epsilon^\kappa.$$

The second equality follows from the fact, that covariant derivative of a metric tensor vanishes. This may be shown using (2) and the definition of the covariant derivative. The term containing Christoffel symbols may be rewritten:

$$\begin{aligned}g_{\nu\alpha}\Gamma^\alpha_{\kappa\mu} + g_{\mu\alpha}\Gamma^\alpha_{\kappa\nu} &= \frac{1}{2}g_{\nu\alpha}g^{\alpha\beta}(\partial_\kappa g_{\beta\mu} + \partial_\mu g_{\kappa\beta} - \partial_\beta g_{\kappa\mu}) +\\ &+\frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(\partial_\kappa g_{\beta\nu} + \partial_\nu g_{\kappa\beta} - \partial_\beta g_{\kappa\nu}) = \partial_\kappa g_{\mu\nu}.\end{aligned}$$

Hence,

$$\delta g_{\mu\nu} = \nabla_\mu\epsilon_\nu + \nabla_\nu\epsilon_\mu. \tag{6}$$

The action variation vanishes, when $\nabla_\mu T^{\mu\nu} = 0$ is satisfied:

$$\delta S = \frac{1}{2}\int d^D x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} = \int d^D x\sqrt{-g}T^{\mu\nu}\nabla_\mu\epsilon_\nu = -\int d^D x\sqrt{-g}\nabla_\mu T^{\mu\nu}\epsilon_\nu.$$

Where in the last equality integration by parts was performed.

2 Conformal Field Theory

The Weyl transformation is a transformation of the form [5]:

$$g_{\mu\nu}(x) \to \Omega(x)g_{\mu\nu}(x).$$

An infinitesimal transformation can be written:

$$g_{\mu\nu}(x) \to g_{\mu\nu} + \omega(x) g_{\mu\nu}(x).$$

The variation of the action yields energy momentum tensor conservation:

$$\delta S := \frac{1}{2} \int d^D x \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} = \frac{1}{2} \int d^D x \sqrt{-g} T^\mu_\mu \omega(x),$$

hence for the Weyl symmetry the energy momentum tensor is traceless:

$$T^\mu{}_\mu = 0.$$

A conformal transformation is transformation of the coordinates, such that metric tensor is rescaled. It is a Weyl transformation performed through the change of basis.
It may be shown, that the group of conformal transformations consists of:

- Spacetime translations: $x^\mu \to x^\mu + \alpha^\mu$
- Lorentz rotations: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}$
- Scale transformations: $x^\mu \to x^\mu + \sigma x^\mu$
- Special conformal transformations: $x^\mu \to x^\mu - 2(b \cdot x) x^\mu + x^2 b^\mu$

Consider infinitesimal transformation $x^\mu = x^\mu - \epsilon^\mu(x')$, the metric tensor variation (6) in the flat space is:

$$\delta g_{\mu\nu} = \omega(x) g_{\mu\nu} = -\partial_\nu \epsilon_\mu - \partial_\mu \epsilon_\nu.$$

Taking the trace of both sides, in 2 dimensions we have:

$$\omega(x) = -\partial_\mu \epsilon^\mu.$$

and the metric variation equation gives:

$$\partial_\nu \epsilon_\mu + \partial_\mu \epsilon_\nu = \partial_\alpha \epsilon^\alpha g_{\mu\nu}, \tag{7}$$

it may be rewritten to the Cauchy-Riemann conditions, which are satisfied by holomorphic functions:

$$\begin{cases} \partial_1 \epsilon_1 = \partial_2 \epsilon_2 \\ \partial_2 \epsilon_1 = -\partial_1 \epsilon_2 \end{cases}$$

Such a structure of space-time translations allows to construct its holomorphic and antiholomorphic parts.
The conformal current of such symmetry is $J_\mu = T_{\mu\nu} \epsilon^\nu$ and it is conserved:

$$\partial^\mu J_\mu(\epsilon) = \big(\partial^\mu T_{\mu\nu}\big) \epsilon^\nu + T_{\mu\nu}(\partial^\mu \epsilon^\nu).$$

The first term vanishes, because of energy-momentum tensor conservation. Using the fact, that energy-momentum tensor is symmetric we may rewrite the second term:

$$\partial^\mu J_\mu(\epsilon) = T_{\mu\nu}(\partial^\mu \epsilon^\nu) = \frac{1}{2} T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) = \frac{1}{2} T^\mu_\mu \partial_\alpha \epsilon^\alpha = 0,$$

where in the second equality (7) has been applied, and in the third equality tracelessness of the energy-momentum tensor used.

3 2D Einstein gravity

In the 2D setup, the only solutions to Einstein equations are vacuum ground state, black holes, and naked singularity. 4D considerations of Hawking radiation may be transferred to 2D theories [4]. Pure gravity is a topological theory in 2D, meaning there are no interesting dynamics. This can be seen, by considering analog of the Einstein-Hilbert action (without cosmological constant) in an arbitrary number of dimensions D:

$$S_{EH} = \int d^D x \sqrt{-g} R.$$

Variation of this action yields the Einstein tensor $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. In the absence of matter, the Einstein equations require this tensor to vanish.
However, in the case of $D = 2$ dimensions this condition is satisfied *identically* because the integrant is a full derivative. All curvature invariants can be expressed by Ricci scalar only because there is only one independent component of Riemann tensor in 2D. The only candidate for Riemann tensor is:

$$R_{abcd} = f(R)(g_{ac}g_{bd} - g_{ad}g_{bc}),$$

where $f(R)$ is a general function of Ricci scalar, the only free parameter in the theory. The term multiplying $f(R)$ is the only tensor, constructed from metric only, possessing appropriate symmetries from section 1.2.
Skew symmetry:

$$R_{bacd} = f(R)(g_{bc}g_{ad} - g_{bd}g_{ac}) = -R_{abcd},$$
$$R_{abdc} = f(R)(g_{ad}g_{bc} - g_{ac}g_{bd}) = -R_{abcd}.$$

Interchange symmetry:

$$R_{cdab} = f(R)(g_{ca}g_{db} - g_{cb}g_{da}) = R_{abcd}$$

First Bianchi identity:

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$
$$g_{ac}g_{bd} - g_{ad}g_{bc} + g_{ad}g_{cb} - g_{ab}g_{cd} + g_{ab}g_{dc} - g_{ac}g_{bd} = 0,$$

where the cancelations are between terms $\{1,6\}, \{2,3\}, \{4,5\}$.
Second Bianchi Identity is satisfied because the Riemann tensor is a function of metrics only:

$$R_{abcd} = R_{abcd}(g_{\alpha\beta}),$$

hence the covariant derivative will act on Riemann tensor as on composite function, covariant derivative of metric can be factorized and the whole expression vanishes.
The symmetries are satisfied and the function $f(R)$ may be uniquely determined by contracting the Riemann tensor twice. First, we get the Ricci tensor:

$$R_{bc} = R^a{}_{bac} = f(R) g^{am}(g_{ma}g_{bc} - g_{mc}g_{ba}) = f(R) g_{bc}$$

and Ricci scalar:

$$R = R^b_b = f(R) g^b_b = 2f(R).$$

Finally we get $f(R) = \frac{R}{2}$ and the Riemann tensor in 2D may be written as:

$$R^\gamma{}_{\nu\alpha\beta} = \frac{R}{2} g^{\gamma\mu}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}). \tag{8}$$

Ricci tensor gives:

$$R_{\mu\nu} = \frac{R}{2} g_{\mu\nu}.$$

From the definition, we see that the Einstein tensor vanishes:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 0.$$

As mentioned before this result shows, that in classical 2D Einstein gravity there are no dynamics- there are no constraints of metric from equations of motion and energy-momentum is equal to zero. For further discussion of Riemannian geometry consult [9].

3.1 Conformal gauge

The main adventage of the conformal 2D gravity is a presence of additional degree of freedom, which lets us [6] express the metric tensor in so called *conformal gauge*:

$$g_{\mu\nu} = e^{\sigma(x)} \delta_{\mu\nu}.$$

A metric tensor may be expressed in such a form by choosing *synchronous frame*- a reference frame in which the time coordinate defines the proper time for all co-moving observers[1]. This lets us set $g_{0i}dx^i$ to zero, which in 2D means that metric tensor is diagonal. Choosing the synchronous frame does not exhaust gauge freedom and we still may perform spatial rotations, this additional degree of freedom lets us set both diagonal components of the metric to be equal e^σ. Calculation of the Ricci scalar is especially simple, using the trick (5):

$$R(x) = -e^{-\sigma(x)}\left(\partial_1^2\sigma(x) + \partial_2^2\sigma(x)\right) = -\partial_\mu\partial^\mu\sigma(x) = -\Delta\sigma(x).$$

The usual CFT approach is to complexify the space with the *conformal coordinates*:

$$z = x^1 + ix^2, \qquad \bar{z} = x^1 - ix^2,$$

metric tensor in the new coordinates takes form

$$g_{\mu\nu} = \begin{bmatrix} e^{\sigma(x)} & 0 \\ 0 & e^{\sigma(x)} \end{bmatrix} \quad \longrightarrow \quad g_{z_\mu z_\nu} = \frac{1}{2}\begin{bmatrix} 0 & e^{\sigma(z,\bar{z})} \\ e^{\sigma(z,\bar{z})} & 0 \end{bmatrix}.$$

[1] There is a good article on Wikipedia describing the procedure of constructing synchronous reference frame- `https://en.wikipedia.org/wiki/Synchronous_frame`. For a more detailed discussion see [3]

It is easy to check the inverse metric is:

$$g^{z_\mu z_\nu} = 2\begin{bmatrix} 0 & e^{-\sigma(z,\bar{z})} \\ e^{-\sigma(z,\bar{z})} & 0 \end{bmatrix}.$$

Ricci scalar in complex coordinates takes form:

$$R(z,\bar{z}) = -4e^{-\sigma(z,\bar{z})}\partial_z\partial_{\bar{z}}\sigma(z,\bar{z}). \tag{9}$$

As it has been shown in the introductory GR section, the energy momentum tensor is conserved. In the complex coordinates the conservation law is equivalent to the pair of equations:

$$\nabla_\mu T^{\mu\nu} = \begin{bmatrix} \nabla_z T^{zz} + \nabla_{\bar{z}} T^{\bar{z}z} \\ \nabla_z T^{z\bar{z}} + \nabla_{\bar{z}} T^{\bar{z}\bar{z}} \end{bmatrix} = 0$$

In the conformal gauge, by applying (4) one gets:

$$\begin{cases} \partial_{\bar{z}} T^{\bar{z}\bar{z}} + 2\partial_{\bar{z}}\sigma T^{\bar{z}\bar{z}} + \partial_z T^{z\bar{z}} + \partial_z\sigma T^{z\bar{z}} = 0 \\ \partial_z T^{z\bar{z}} + \partial_z\sigma T^{z\bar{z}} = 0 \end{cases}$$

3.2 Conformal anomaly

In the flat space, the CFT structure of a theory implies a vanishing trace of the energy-momentum tensor. This is not the case in curved space, instead, the trace is characterized by the *conformal anomaly equation* which is of great importance in the quantum case:

$$T^\mu{}_\mu = \alpha R(x).$$

Recall, the form of Ricci scalar in complex coordinates (9). We find the explicit form of components of the energy-momentum tensor:

$$\begin{aligned} T^z{}_z + T^{\bar{z}}{}_{\bar{z}} &= -4\alpha e^{-\sigma}\partial_z\partial_{\bar{z}}\sigma \\ 2g^{z\bar{z}}T_{z\bar{z}} = 4e^{-\sigma}T_{z\bar{z}} &= -4\alpha e^{-\sigma}\partial_z\partial_{\bar{z}}\sigma \\ T_{z\bar{z}} &= -\alpha\partial_z\partial_{\bar{z}}\sigma \end{aligned}$$

Continuity equation for component z can be written as:

$$\partial_{\bar{z}} T_{zz} + e^\sigma \partial_z\Big(e^{-\sigma}T_{z\bar{z}}\Big) = 0,$$

while in the flat space the continuity equation in complex coordinates gives: $\partial_z T_{\bar{z}\bar{z}} = 0$ and $\partial_{\bar{z}} T_{zz} = 0$.

The continuity equation implies existence of holomorphic structure of the energy-momentum tensor. We may define holomorphic pseudotensor:

$$\partial_{\bar{z}} T := \partial_{\bar{z}}\Big[T_{zz} - \frac{\alpha}{2}\Big(-(\partial_z\sigma)^2 + 2\partial_z^2\sigma\Big)\Big] = 0,$$

antiholomorphic pseudotensor:

$$\partial_z \bar{T} := \partial_z\Big[T_{\bar{z}\bar{z}} - \frac{\alpha}{2}\Big(-(\partial_{\bar{z}}\sigma)^2 + 2\partial_{\bar{z}}^2\sigma\Big)\Big] = 0.$$

Notice these objects are not tensors, their coordinate transformation is anomalous.

4 Conformal quantum gravity

Years of research in the field of quantum gravity recognized numerous problems. The Einstein-Hilbert in the standard form is not power-counting renormalizable at one loop, including mass fields [7]. Various solutions to this theoretical problem have been proposed, e.g. including higher-order Riemann tensor contractions[2]. The discussion about renormalizability of the constructed theory lays beyond the scope of this paper.
The attempts of quantizing gravity are usually using the Feynman path integral approach. The definition of such objects is usually troublesome. However, as it will be shown, the anomaly equation provides great control over the partition function.
The full form of the anomaly equation is given by:

$$T^{\mu}{}_{\mu} = -\frac{c}{12}R(x),$$

where c is the central charge of the theory.
Consider now an infinitesimal conformal transformation of the metric:

$$g_{\mu\nu} \longrightarrow (1 + \delta\sigma(x))g_{\mu\nu}.$$

As before the variation of the action is of form

$$\delta S = \frac{1}{4\pi}\int \sqrt{-g}T^{\mu}{}_{\mu}\delta\sigma(x)d^2x,$$

The partition function is given by:

$$Z[g] = \int e^{-S[g,\phi]}D[\phi].$$

We assume the metric has been conformally transformed from some reference metric, which is not necessarily flat. The conformal gauge is abandoned from now on:

$$g_{\mu\nu}(x) = e^{\sigma(x)}\hat{g}_{\mu\nu}(x).$$

From this form we get the Ricci scalar associated with the metric g:

$$\sqrt{g}R(x) = \sqrt{\hat{g}}\Big(\hat{R}(x) - \Delta_{\hat{g}}\sigma(x)\Big).$$

Form of the partition function may be evaluated from:

$$\delta \log Z[e^{\sigma}\hat{g}] = Z^{-1}Z\delta S = \frac{c}{48\pi}\int dx^2\sqrt{\hat{g}}\Big(\hat{R}(x) - \Delta_{\hat{g}}\sigma(x)\Big)\delta\sigma.$$

[2]This apparent solution makes gravity renormalizable, however, predicts the existence of ghost particles. In some models, ghosts have been avoided, as in Horava gravity, where one abandons Lorentz invariance in high energies.

Thanks to the chain rule, and assuming vanishing boundry terms one gets Functional differential equation:

$$\frac{\delta S}{\delta\sigma(x)} = \frac{c}{48\pi}\Big(\hat{R}(x) - \Delta_{\hat{g}}\sigma(x)\Big).$$

The solution to this equation may be found by simple integration, as RHS is independent of terms containing field $\sigma(x)$ ($\Delta\sigma$ is an independent field). Keeping in mind the initial condition we obtain:

$$S = \frac{c}{48\pi}\int dx^2\sqrt{\hat{g}}\Big(\hat{R}(x)\sigma(x) - \sigma(x)\Delta_{\hat{g}}\sigma(x)\Big) + S[\hat{g}].$$

Hence the partition function:

$$Z[e^{\sigma}\hat{g}] = \exp\Big[\frac{c}{48\pi}\int dx^2\sqrt{\hat{g}}\Big(\hat{R}(x)\sigma(x) + \frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma(x)\Big)\Big]Z[\hat{g}].$$

Using the anomaly equation we have constructed a partition function, in which the integrand is called *Liouville Lagrangian*. Interestingly, we have expressed partition function of an ambiguous, conformaly transformed metric tensor with partition function of the reference metric.

4.1 Liouville equation

The action constructed in previous section may include cosmological term, which is simply a volume integral in conformaly curved space-time with Λ being the cosmological constant:

$$S_L[\sigma,\hat{g}] = \frac{1}{2\pi}\int dx^2\sqrt{\hat{g}}\Big(\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma(x) + \hat{R}(x)\sigma(x) + \Lambda e^{\sigma(x)}\Big). \tag{10}$$

Euler-Lagrange equations follow:

$$-\Delta_{\hat{g}}\sigma(x) + \hat{R}(x) + \Lambda e^{\sigma(x)} = 0.$$

Recall the form of Ricci scalar in the chosen transformation:

$$\sqrt{\hat{g}}\Big(\hat{R}(x) - \Delta_{\hat{g}}\sigma(x)\Big) = \sqrt{g}R.$$

E-L equations may be rewritten to:

$$\hat{R}(x) + \Lambda = 0.$$

Thus the classical solutions to the Liouville are spaces with constant curvature. For a flat reference metric we get the *Liouville equation*:

$$-4\partial_z\partial_{\bar{z}}\sigma(z,\bar{z}) + \Lambda e^{\sigma} = 0. \tag{11}$$

4.2 Liouville gravity as CFT

The construction of the Liouville theory assumed conformal invariance, however, one may assure himself of this symmetry. Via direct variation of the action we get the energy-momentum tensor:

$$T_{\mu\nu} = -\partial_\mu\sigma\partial_\nu\sigma + \hat{g}_{\mu\nu}\Big(\frac{1}{2}(\partial\sigma)^2 + \Lambda e^{\sigma}\Big) + 2\Big(\partial_\mu\partial_\nu\sigma - \hat{g}_{\mu\nu}\partial^2\sigma\Big).$$

Its trace is:

$$T^{\mu}{}_{\mu} = 2\Big(\Lambda e^{\sigma} - \partial^2\sigma\Big) = 0.$$

And disappears, given the Liouville equation is satisfied. We have seen this property is characteristic for CFTs. The holomorphic structure of energy-momentum tensor is:

$$T_{zz} = t(z) = -\partial_z\sigma\partial_z\sigma + 2\partial_z^2\sigma,$$
$$T_{\bar{z}\bar{z}} = \bar{t}(\bar{z}) = -\partial_{\bar{z}}\sigma\partial_z\sigma + 2\partial_{\bar{\partial}z}^2\sigma.$$

And finally the Liouville equation is itself invariant under simultaneous transformations:

$$z \longrightarrow w(z),$$
$$\bar{z} \longrightarrow \bar{w}(\bar{z}),$$
$$\sigma(z,\bar{z}) \longrightarrow \sigma(z,\bar{z}) - \log\Big(\frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\Big).$$

We check it by transforming (11):

$$-4\partial_{z'}\partial_{\bar{z}'}\sigma'(z,\bar{z}) + \Lambda e^{\sigma'} = 0,$$
$$-4\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\partial_z\partial_{\bar{z}}\sigma + 4\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}}\partial_z\partial_{\bar{z}}\log\left(\frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}\right) + \Lambda e^{\sigma}\frac{dz}{dw}\frac{d\bar{z}}{d\bar{w}} = 0.$$

The term with logarithm may be rewritten $\log(\frac{dw}{dz}\frac{d\bar{w}}{d\bar{z}}) = \log\frac{dw}{dz} + \log\frac{d\bar{w}}{d\bar{z}}$. Acting with the operator $\partial_z\partial_{\bar{z}}$ on a function $f(z,\bar{z}) = g(z) + h(\bar{z})$ gives zero, hence the second term vanishes. Dividing both sides by $\frac{d\bar{z}}{d\bar{w}}$ restores Liouville equation.

4.3 Correlation function

In the path integral approach to the Quantum Field Theory an object of particular interest is the correlation function:

$$\langle\mathcal{O}_{j_1}(x_1),\ldots,\mathcal{O}_{j_n}(x_n)\rangle = Z^{-1}\int D[\phi(x)]\mathcal{O}_{j_1}(x_1)\ldots\mathcal{O}_{j_n}(x_n)e^{-S[\phi(x)]},$$

where the $\mathcal{O}_j(x)$ is a basis vector of space of the composite fields:

$$\mathcal{F} := span\{\phi(x)^k, \phi^k(x)\partial_\mu\phi, \phi^k(x)\partial_\mu\phi\partial_\nu\phi,\ldots\},$$

consisting of powers of fields, and its derivatives. Scalar product of the two vectors in such a space is a integration of the two fields over whole space-time. In quantum gravity we include metric as independent field and denote:

$$\langle \mathcal{O}^{[g]}_{j_1}(x_1), \ldots, \mathcal{O}^{[g]}_{j_n}(x_n)\rangle = Z^{-1} \int D[\phi]D[g]\mathcal{O}^{[g]}_{j_1}(x_1)\ldots\mathcal{O}^{[g]}_{j_n}(x_n)e^{-S[\phi,g]}, \quad (12)$$

where $Z = \int D[\phi]D[g]\exp(-S[\phi, g])$. The meaning of the measure $D[\phi]$ in the flat-space QFT may be realised by rewriting (12) in momentum space divided into "small boxes". One may then operate on the path integral as a limit of the ratio of finite dimensional integrals.
When it comes to gravity, gauge ambiguity must be taken into account, when integrating over the space of metric tensors. Therefore one may expect the form of the measure:

$$D[g] = \frac{D[g_{\mu\nu}]}{D[\epsilon]}, \quad (13)$$

where $D[g_{\mu\nu}]$ is a local functional measure for tensor fields $g_{\mu\nu}(x)$, while $D[\epsilon]$ is a measure on the space of diffeomorphisms vector fields $\epsilon^{\mu}(x)$. In CFT the variation of metric tensor, under conformal and coordinate transformation gives us:

$$\delta g_{\mu\nu} = g_{\mu\nu}\delta\sigma(x) + \nabla_{\mu}\epsilon_{\nu}(x) + \nabla_{\nu}\epsilon_{\mu}(x). \quad (14)$$

Introduced by Polyakov [8] ultralocal metrics on spaces of rank 2 tensors $g_{\mu\nu}$, and vector fields ϵ^{μ} let us evaluate the measure in path integral.

$$||\delta g_{\mu\nu}||^2 = \int \sqrt{g}d^2x\Big(g^{\mu\alpha}g^{\nu\beta} + Cg^{\mu\nu}g^{\alpha\beta}\Big)\delta g_{\mu\nu}\delta g_{\alpha\beta} \quad (15)$$

$$||\epsilon||^2 = \int \sqrt{g}d^2x\ g_{\mu\nu}\epsilon^{\mu}\epsilon^{\nu}. \quad (16)$$

One may now shift $\delta\sigma(x)$ in (14), so that $\delta\sigma + \nabla_{\lambda}\epsilon^{\lambda} \to \delta\sigma$. Hence:

$$\delta g_{\mu\nu} = g_{\mu\nu}\delta\sigma(x) + \varepsilon_{\mu\nu}, \qquad \varepsilon_{\mu\nu} := \nabla_{\mu}\epsilon_{\nu} + \nabla_{\nu}\epsilon_{\mu} - g_{\mu\nu}\nabla_{\lambda}\epsilon^{\lambda}.$$

The measure (15) takes form:

$$\begin{aligned} ||\delta g_{\mu\nu}||^2 = \int \sqrt{g}d^2x\Big(&g^{\mu\alpha}g^{\nu\beta}(g_{\mu\nu}\delta\sigma + \varepsilon_{\mu\nu})(g_{\alpha\beta}\delta\sigma + \varepsilon_{\alpha\beta}) + \\ &+ Cg^{\mu\nu}g^{\alpha\beta}(g_{\mu\nu}\delta\sigma + \varepsilon_{\mu\nu})(g_{\alpha\beta}\delta\sigma + \varepsilon_{\alpha\beta})\Big) \\ = \int \sqrt{g}d^2x\Big(&g^{\mu}_{\mu}(\delta\sigma)^2 + 2\varepsilon^{\mu}_{\mu}\delta\sigma + \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} + C(g^{\mu}_{\mu})^2 + 2g^{\mu}_{\mu}\varepsilon^{\alpha}_{\alpha}\delta\sigma + (\varepsilon^{\mu}_{\mu})^2\Big). \end{aligned}$$

Trace of the $\varepsilon_{\mu\nu}$ tensor is given by:

$$\varepsilon^{\mu}_{\mu} = (2 - g^{\mu}_{\mu})\nabla_{\lambda}\epsilon^{\lambda},$$

therefore in 2 dimensions it vanishes. Applying this result one gets:

$$||\delta g_{\mu\nu}||^2 = \int \sqrt{g}d^2x\Big((4C + 2)\delta\sigma^2 + \varepsilon_{\mu\nu}\varepsilon^{\mu\nu}\Big).$$

Notice, the conformal and diffeomorphism measures are independently summed, hence the integration may be performed independently over conformal and gauge fields. This let's us rewrite (13):

$$D[g] = \frac{D[g_{\mu\nu}]}{D[\epsilon]} = det[\varepsilon]D[\sigma].$$

Using the Fadeev-Poppov formalism gives us a way to evaluate the determinant and interpret it as new, anticommuting gauge ghosts, defined as:

$$det[\varepsilon] = Z_{ghost}[g] = \int D[B,C]e^{-S_{ghost}[B,C,g]}.$$

Ghost action S_{ghost} consists of tensor field $B_{\mu\nu}$ and vector field C^{μ}:

$$A_{ghost}[B,C,g] = \frac{1}{2\pi}\int \sqrt{g}d^2x\ B_{\mu\nu}\nabla^{\mu}C^{\nu}.$$

We have built a theoretical framework, which allows the exact calculation of the correlation function. In particular, the gauge ambiguity is not problematic anymore, it has been reduced to the path integral over simpler $D[\sigma]$ space. Further development of the theory would include explicit calculation of the correlation function, for a given set of the composite fields.
Reparemetrizing the Liouville Lagrangian (10):

$$S = \int d^2z\left(\frac{1}{4\pi}(\partial_a\phi)^2 + \mu e^{2b\phi(z,\bar{z})}\right),$$

for upper half plane we have bulk one-point function of a field [10] $V_\alpha(z,\bar{z}) = e^{2\alpha\phi(z,\bar{z})}$:

$$\langle V_\alpha(z,\bar{z})\rangle = \frac{U_\sigma(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}},$$

where Δ_σ is the conformal dimension of a parameter $\sigma(\Lambda)$ and

$$U_\sigma(\alpha) = \frac{2}{b}\left(\pi\Lambda\gamma(\beta^2)\right)^{\frac{Q-2\alpha}{2b}}\Gamma(2b\alpha-b^2)\Gamma(\frac{2\alpha}{b}-\frac{1}{b^2}-1)\cos\left[\pi(2\alpha-Q)(2\sigma-Q)\right].$$

$U_\sigma(\alpha)$ has a reflection property:

$$U_\sigma(\alpha) = S(\alpha)U_\sigma(Q-\alpha),$$

where $S(\alpha)$ is bulk reflection amplitude:

$$S(\alpha) = \frac{(\pi\Lambda\gamma(b^2))^{(Q-2\alpha)/b}}{b^2}\frac{\gamma(2\alpha b-b^2)}{\gamma(2-2\alpha/b+1/b^2}.$$

The observables are invariant with respect to the duality transformation $b \to \frac{1}{b}$. Reflection property for the boundary two-point function takes form of a unitarity condition. The three-point function is invariant under cyclic permutation of the fields.

5 Summary

In the paper construction of 2D Liouville gravity has been shown. The derivations have been as explicit as possible, introductory sections are supplied. Exceptionally simple form of gravity with conformal symmetry in two dimensions is a valuable framework, in which the crucial aspects of the theory are not overlooked in tedious calculations.
In the future work one may proceed with the quantization, include matter fields and examine interactions in the theory. Interesting behaviour may occur, while considering $\sigma(x) \sim \frac{1}{x}$, Laplacian of such function is a dirac delta and connection to the Green's functions may be found.

6 Appendix

6.1 Covariant derivative transformation

In order to show that the covariant derivative of a vector field (3) transforms as a tensor, we first state how does rank (m, n) tensor transform $T \to T'$ under general coordinate transformation $x \to x'$:

$$T'^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_n}}{\partial x'^{\nu_n}} T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}. \quad (17)$$

For example, partial derivative of a scalar field transforms as a rank (1,0) tensor:

$$\frac{\partial \phi}{\partial x^\mu} \to \frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}.$$

However, partial derivative of a vector field does not transform in the described way:

$$\frac{\partial v^\nu}{\partial x^\mu} \to \frac{\partial v'^\nu}{\partial x'^\mu} = \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial}{\partial x^\beta} \Big(\frac{\partial x'^\nu}{\partial x^\alpha} v^\alpha \Big) = \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\beta \partial x^\alpha} v^\alpha + \frac{\partial x^\beta}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\alpha} \frac{\partial v^\alpha}{\partial x^\beta}. \quad (18)$$

The second term is what we expect from a tensor transformation, but we get and additional term linear with respect to v^α.
We show here, that covariant derivative of a vector field (3) indeed transforms as rank (2,0) tensor. The partial derivative term in (3) transforms as (18), hence we expect, that the Christoffel symbols also do not transform as tensors. Consider a transformation:

$$\Gamma^\nu{}_{\mu\rho} \to \Gamma'^\nu{}_{\mu\rho} = \frac{1}{2} g'^{\nu\sigma} \Big(\frac{\partial g'_{\mu\sigma}}{\partial x'^\rho} + \frac{\partial g'_{\rho\sigma}}{\partial x'^\mu} - \frac{\partial g'_{\mu\rho}}{\partial x'^\sigma} \Big),$$

notice that both metric tensor and partial derivatives must be transformed. We get:

$$\Gamma'^\nu{}_{\mu\rho} = \frac{1}{2} \frac{\partial x'^\nu}{\partial x^\alpha} \frac{\partial x'^\sigma}{\partial x^\beta} g^{\alpha\beta} \Bigg(\frac{\partial x^\gamma}{\partial x'^\rho} \frac{\partial}{\partial x^\gamma} \Big(\frac{\partial x^\delta}{\partial x'^\mu} \frac{\partial x^\xi}{\partial x'^\sigma} g_{\delta\xi} \Big) +$$
$$+ \frac{\partial x^\gamma}{\partial x'^\rho} \frac{\partial}{\partial x^\gamma} \Big(\frac{\partial x^\delta}{\partial x'^\rho} \frac{\partial x^\xi}{\partial x'^\sigma} g_{\delta\xi} \Big) - \frac{\partial x^\gamma}{\partial x'^\sigma} \frac{\partial}{\partial x^\gamma} \Big(\frac{\partial x^\delta}{\partial x'^\mu} \frac{\partial x^\xi}{\partial x'^\rho} g_{\delta\xi} \Big) \Bigg).$$

To make the calculation clearer we evaluate each of the above terms separately. The first term:

$$\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}g^{\alpha\beta}\frac{\partial x^\gamma}{\partial x'^\rho}\frac{\partial}{\partial x^\gamma}\Big(\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\sigma}g_{\delta\xi}\Big) =$$

$$= \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\rho}g^{\alpha\beta}\Bigg(\frac{\partial^2 x^\delta}{\partial x^\gamma \partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\sigma}g_{\delta\xi} + \frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial^2 x^\xi}{\partial x^\gamma \partial x'^\sigma}g_{\delta\xi} + \frac{\partial x^\delta}{\partial x'^\sigma}\frac{\partial g_{\delta\xi}}{\partial x^\gamma}\Bigg) =$$

$$= \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\rho \partial x'^\mu} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2 x^\xi}{\partial x'^\rho \partial x^\beta}g^{\alpha\beta}g_{\delta\xi}\frac{\partial x^\delta}{\partial x'^\mu} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}g^{\alpha\xi}\frac{\partial g_{\delta\xi}}{\partial x'^\rho}.$$

The second term:

$$\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}g^{\alpha\beta}\frac{\partial x^\gamma}{\partial x'^\rho}\frac{\partial}{\partial x^\gamma}\Big(\frac{\partial x^\delta}{\partial x'^\rho}\frac{\partial x^\xi}{\partial x'^\sigma}g_{\delta\xi}\Big) = \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\sigma}g^{\alpha\beta}g_{\delta\xi}\frac{\partial^2 x^\delta}{\partial x^\gamma \partial x'^\rho}+$$

$$+\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\rho}g^{\alpha\beta}g_{\delta\xi}\frac{\partial^2 x^\xi}{\partial x^\gamma \partial x'^\sigma} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}\frac{\partial x^\gamma}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\gamma}g^{\alpha\beta}\frac{\partial g_{\delta\xi}}{\partial x^\gamma} =$$

$$= \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\delta}\frac{\partial^2 x^\delta}{\partial x'^\mu \partial x'^\gamma} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\rho}\frac{\partial^2 x^\xi}{\partial x'^\mu \partial x^\beta}g^{\alpha\beta}g_{\delta\xi} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\rho}g^{\alpha\xi}\frac{\partial g_{\delta\xi}}{\partial x'^\mu}.$$

The third term:

$$-\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x'^\sigma}{\partial x^\beta}g^{\alpha\beta}\frac{\partial x^\gamma}{\partial x'^\sigma}\frac{\partial}{\partial x^\gamma}\Big(\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}g_{\delta\xi}\Big) =$$

$$= -\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}g^{\alpha\gamma}\Bigg(\frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x^\gamma}\frac{\partial x^\xi}{\partial x'^\rho}g_{\delta\xi} + \frac{\partial^2 x^\xi}{\partial x^\gamma \partial x'^\rho}\frac{\partial x^\delta}{\partial x'^\mu}g_{\delta\xi} + \frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}\frac{\partial g_{\delta\xi}}{\partial x^\gamma}\Bigg) =$$

$$= -\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\xi}{\partial x'^\rho}g^{\alpha\gamma}g_{\delta\xi}\frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x^\gamma} - \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}g^{\alpha\gamma}g_{\delta\xi}\frac{\partial^2 x^\xi}{\partial x^\gamma \partial x'^\rho}+$$

$$-\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}g^{\alpha\gamma}\frac{\partial g_{\delta\xi}}{\partial x^\gamma}.$$

Adding the three terms gives us:

$$\Gamma'^\nu{}_{\mu\rho} = \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\rho \partial x'^\mu} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}g^{\alpha\xi}\frac{\partial g_{\delta\xi}}{\partial x'^\rho} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\rho}g^{\alpha\xi}\frac{\partial g_{\delta\xi}}{\partial x'^\mu}+$$

$$-\frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}g^{\alpha\gamma}\frac{\partial g_{\delta\xi}}{\partial x^\gamma} =$$

$$= \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\rho \partial x'^\mu} + \frac{1}{2}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}g^{\alpha\gamma}\Bigg(-\frac{\partial g_{\delta\xi}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\xi} + \frac{\partial g\gamma\xi}{\partial x^\delta}\Bigg) =$$

$$= \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\rho \partial x'^\mu} + \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}\Gamma^\alpha{}_{\delta\xi}.$$

As expected, the Christoffel symbols are not tensors. Because of the first term in above expression transformation rule (17) does not hold.
We may combine the above result with (18) to get:

$$(\nabla_\mu v^\nu)' = \partial'_\mu v'^\nu + \Gamma'^\nu{}_{\mu\rho}v'^\rho = \frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\beta \partial x^\alpha}v^\alpha + \qquad (19)$$

$$+\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial v^\alpha}{\partial x^\beta} + \Bigg(\frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\rho \partial x'^\mu} + \frac{\partial x'^\nu}{\partial x^\alpha}\frac{\partial x^\delta}{\partial x'^\mu}\frac{\partial x^\xi}{\partial x'^\rho}\Gamma^\alpha{}_{\delta\xi}\Bigg)\frac{\partial x'^\rho}{\partial x^\beta}v^\beta. \qquad (20)$$

We show now, that the first and third terms cancels. We may rewrite:

$$\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\rho}\partial x'^{\mu}}\frac{\partial x'^{\rho}}{\partial x^{\beta}}v^{\beta} = \frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta}\partial x'^{\mu}}v^{\beta}.$$

From the identity $\frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$ we get:

$$\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial x'^{\alpha}}{\partial x^{\nu}}\right) = 0$$

$$\frac{\partial x'^{\alpha}}{\partial x^{\nu}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x'^{\alpha}} = -\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial^2 x'^{\alpha}}{\partial x^{\beta}\partial x^{\nu}}$$

$$\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x'^{\kappa}} = -\frac{\partial x^{\mu}}{\partial x'^{\alpha}}\frac{\partial^2 x'^{\alpha}}{\partial x^{\beta}\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial x'^{\kappa}}.$$

Using this result we evaluate:

$$\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta}\partial x'^{\mu}}v^{\beta} = -\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial^2 x'^{\rho}}{\partial x^{\beta}\partial x^{\sigma}}\frac{\partial x^{\alpha}}{\partial x'^{\rho}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}v^{\beta} =$$

$$= -\frac{\partial^2 x'^{\nu}}{\partial x^{\beta}\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}v^{\beta} = -\frac{\partial^2 x'^{\nu}}{\partial x^{\beta}\partial x'^{\mu}}v^{\beta}.$$

Therefore the first and third term in (20) cancels, leaving:

$$(\nabla_{\mu}v^{\nu})' = \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial v^{\alpha}}{\partial x^{\beta}} + \frac{\partial x'^{\rho}}{\partial x^{\beta}}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial x^{\delta}}{\partial x'^{\mu}}\frac{\partial x^{\xi}}{\partial x'^{\rho}}\Gamma^{\alpha}{}_{\delta\xi}v^{\beta} =$$

$$= \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial v^{\alpha}}{\partial x^{\beta}} + \frac{\partial x'^{\nu}}{\partial x^{\alpha}}\frac{\partial x^{\delta}}{\partial x'^{\mu}}\Gamma^{\alpha}{}_{\delta\beta}v^{\beta} = \frac{\partial x^{\beta}}{\partial x'^{\mu}}\frac{\partial x'^{\nu}}{\partial x^{\alpha}}\left(\frac{\partial v^{\alpha}}{\partial x^{\beta}} + \Gamma^{\alpha}{}_{\beta\gamma}v^{\gamma}\right),$$

which is exactly the form of transformation we were expecting.

References

[1] Misner, C. W.; Thorne, K. S.; and Wheeler, J. A *Geodesic Deviation and the Riemann Curvature Tensor* §8.7 in Gravitation. San Francisco: W. H. Freeman, pp. 218-224, 1973.

[2] Sean M. Carroll. *Lecture Notes on General Relativity*. University of California, Santa Barbara, 1997.

[3] L. D. Landau, E. M. Lifshitz *The Classical Theory of Fields*. Institute of Physical Problems, Academy of Sciences of the U.S.S.R.

[4] Thomas Strobi *Gravity in Two Spacetime Dimensions*, Habilitation thesis RWTH Aachen, May 1999

[5] A.N. Schellekens *Conformal Field Theory*, January 2017

[6] Alexei Zamolodchikov, Alexander Zamolodchikov *Lectures on Liouville Theory and Matrix Models*,

[7] Hooft, G.; Veltman, M.J.G. *One-loop divergencies in the theory of gravitation*, Ann. Inst. H. Poincare Phys. Theor. 1974, A20, 69.

[8] A. M. Polyakov, Mod. Phys. Leu. Al (1987) 899

[9] Mikio Nakahara *Geometry, Topology and Physics* Institute of Physics Publishing, Bristol and Philadelpha, 2003

[10] Ivan K. Kostov, Benedicte Ponsot, Didina Serban *Boundary Liouville Theory and 2D Quantum Gravity* Nucl.Phys. B683 (2004) 309-362